Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 5, pp. 793–804, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00104-9

THIRD-ORDER RESONANCE IN A HAMILTONIAN SYSTEM WITH ONE DEGREE OF FREEDOM†

A. P. MARKEYEV

Moscow

(*Received* 28 *October* 1993)

Non-linear oscillations of a nearly integrable Hamiltonian system with one degree of freedom, which is 2π-periodic in t, are investigated in a small finite neighbourhood of equilibrium. The Hamiltonian is assumed to be analytic, the linearized system is stable, and its characteristic exponents $\pm i\nu$ are purely imaginary, where 3ν is an integer. The equilibrium position of such a system is generally unstable and six trajectories exist that asymptotically approach the equilibrium point as $t \to \pm\infty$ [1, 2].

It is shown that for most initial data the motion is quasi-periodic in the neighbourhood of the equilibrium. The existence of stable 6π-periodic motions near an unstable equilibrium position is established. It is shown that, irrespective of instability, trajectories beginning sufficiently close to an equilibrium point will always remain at a finite distance from it. An estimate is obtained for this distance. The stochastic nature of the motion near trajectories asymptotic to the equilibrium point is discussed.

1. STATEMENT OF THE PROBLEM

We shall study the motion of a Hamiltonian system with one degree of freedom, the Hamiltonian being

$$H = H^{(0)}(x,y)+\varepsilon H^{(1)}(x,y,t)+\varepsilon^2 H^{(2)}(x,y,t)+\dots \tag{1.1}$$

where x and y are the coordinate and the momentum, ε is a small parameter $(0<\varepsilon \ll 1)$ and H is an analytic function of ε, continuous and 2π-periodic in t. The origin $x=y=0$ is an equilibrium position of the system and H is analytic in x, y in the neighbourhood of the origin. It is also assumed that, apart from ε, H also depends on one or more other parameters.

Let us assume that the system, linearized with respect to x and y, is stable in Lyapunov's sense, and its characteristic exponents $\pm i\nu$ are purely imaginary (where ν is a real number other than zero). We shall assume that 3ν is an integer, i.e. the system has a third-order resonance. Then, in the general case, the introduction of arbitrarily small non-linear terms into the equations of motion will disturb the linear stability of the equilibrium position [1].

Instability of a Hamiltonian system at resonance is closely related to the existence of trajectories that are asymptotic as $t \to \pm\infty$ to those in unperturbed motion [3]. It has been shown [2] that a system with one degree of freedom having a third-order resonance has six asymptotic trajectories, three of which tend to those in unperturbed motion as $t \to +\infty$ and three as $t \to -\infty$.

Consider a system with Hamiltonian

†*Prikl. Mat. Mekh.* Vol. 58, No. 5, pp. 37–48, 1994.

$$H=-\frac{1}{6}(x^2+y^2)-\frac{\sqrt{2}}{4}\left[x(x^2-3y^2)\cos t-y(y^2-3x^2)\sin t\right]-\frac{1}{4}(x^2+y^2)^2 \tag{1.2}$$

This system has a resonance $3\nu=1$.

Changing to a rotating system, of coordinates x_1, x_2, where

$$x=\cos\frac{t}{3}x_1-\sin\frac{t}{3}x_2,\quad y=\sin\frac{t}{3}x_1+\cos\frac{t}{3}x_2$$

and introducing canonically conjugate polar coordinates θ, ρ by the relations

$$x_1=\sqrt{2\rho}\cos\theta,\quad x_2=\sqrt{2\rho}\sin\theta \tag{1.3}$$

we obtain a new Hamiltonian

$$H=\rho^2+\rho^{3/2}\cos 3\theta \tag{1.4}$$

If we ignore fourth-order terms in x and y in (1.2), the Hamiltonian (1.4) becomes $H=\rho^{3/2}\cos 3\theta$. A system with this Hamiltonian in coordinates x_1 and x_2 has the phase portrait shown in Fig. 1. The only trajectories asymptotic to the origin are those with constant values of the angle θ: $\theta=\pi/6+n\pi/3$ $(n=1, 2, \ldots, 6)$. In that situation

$$\rho(t)=4\rho(0)\left[2-(-1)^n 3\rho^{1/2}(0)t\right]^{-2}$$

If n is odd (even), the trajectories are asymptotic to the point $x_1=x_2=0$ as $t\to+\infty$ $(t\to-\infty)$. The origin is unstable. If the initial values of θ differ from $\pi/2$, $7\pi/6$ or $11\pi/6$, then for arbitrarily small initial data $\rho(0)\neq 0$ the trajectories move away from the origin as t increases, reaching arbitrarily large distances.

If the fourth-order terms in (1.2) are included, the system has the phase portrait illustrated in Fig. 2. In a sufficiently small neighbourhood of the origin, as before, there are six asymptotic trajectories. However, in a finite neighbourhood these trajectories are not just asymptotic, but homoclinic doubly asymptotic [4, p. 335]: the trajectories approach the origin both as $t\to+\infty$

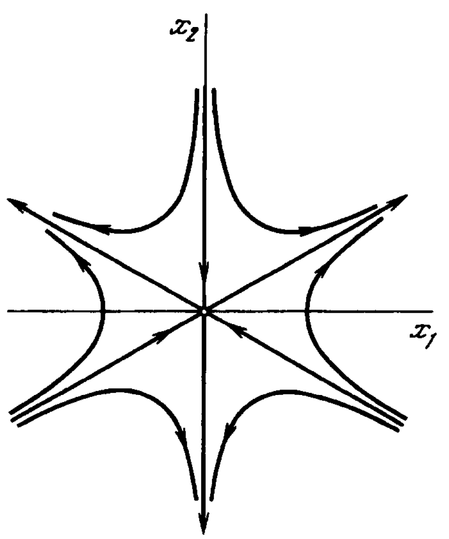

Fig. 1.

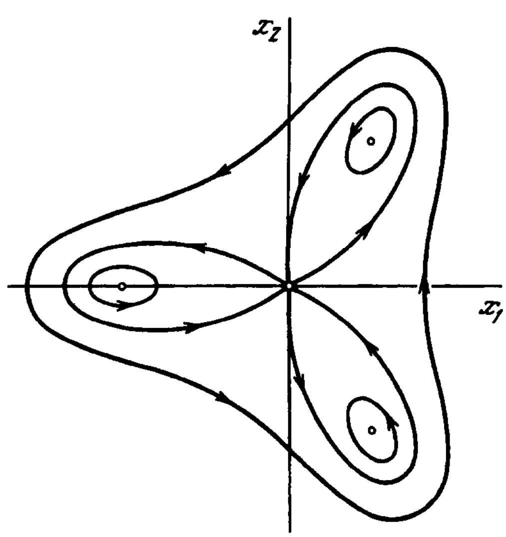

Fig. 2.

and as $t \to -\infty$. Because of the fourth-order terms in (1.2), the six asymptotic trajectories of Fig. 1 merge two by two into the three doubly asymptotic homoclinic trajectories of Fig. 2, and the motion occurs in a bounded neighbourhood of the origin.

Consideration of this example raises several questions about the behaviour of the Hamiltonian at third-order resonance. For example, will the trajectories of the system remain in a bounded neighbourhood of the origin for all t if the Hamiltonian (1.2) is modified by adding small terms periodic in t? Under those conditions, what can one say about non-linear oscillations in a small finite (but not infinitely small) neighbourhood of the origin?

The present paper is devoted to a study of these and some other questions for systems with the Hamiltonian (1.1).

2. PRELIMINARY TRANSFORMATION OF THE HAMILTONIAN

The functions $H^{(i)}$ occurring in expansion (1.1) are represented by the series

$$H^{(i)} = \sum_{k=2}^{\infty} H_k^{(i)}(x,y,t) \tag{2.1}$$

where $H_k^{(i)}$ is a form of degree k in x and y.

The variables x and y may be chosen in such a way that all the second-order forms in (1.1) are in normal form $\nu(x^2+y^2)/2$. This choice of x and y may be achieved by applying a linear canonical 2π-periodic (in t) change of variables [1]. For sufficiently small ε this change of variable will be analytic in ε. In addition, we may assume without loss of generality that x and y have been chosen in such a way that the series expansions (2.1) of $H^{(0)}$, do not contain third- and fifth-degree forms, while the fourth-degree form is a function of the sum x^2+y^2 only. This may be done by applying a non-linear normalizing transformation constructed by the Deprit–Hori method [5] or the classical Birkhoff method [6].

If x and y have been chosen as specified, then, after the canonical transformation $x_* = \varepsilon^{-1}x$, $y_* = \varepsilon^{-1}y$, the Hamiltonian (1.1) becomes

$$H = \nu(x_*^2+y_*^2)/2 + \varepsilon^2\left[H_3^{(1)}(x_*,y_*,t) + c(x_*^2+y_*^2)^2/4\right] + \\ +\varepsilon^3\left[H_3^{(2)}(x_*,y_*,t) + H_4^{(1)}(x_*,y_*,t)\right] + O(\varepsilon^4) \tag{2.2}$$

where c is a constant determined in the non-linear normalization of $H^{(0)}$. We shall assume that $c \neq 0$ and also that the averages of the functions $H_3^{(1)}$, $H_3^{(2)}$, $H_4^{(1)}$ over the explicitly occurring time are zero. Otherwise, these averages may be included in $H^{(0)}$, in which case the only further change occurs in the constant c in (2.2)—but this change is only by a quantity of the order of ε.

Let $3\nu = N$. Applying a Birkhoff-type non-linear canonical change of variables x_*, $y_* \to \xi$, η, where ξ and η are 2π-periodic in t, we can simplify the third degree form $H_3^{(1)}$ in (2.2). This transformation leaves terms of order ε^3 in the Hamiltonian (2.2) unchanged, and we obtain [1]

$$H = \nu(\xi^2+\eta^2)/2 + \varepsilon^2\left[c(\xi^2+\eta^2)^2/4 + (\kappa_1 \sin Nt + \kappa_2 \cos Nt)(\eta^3 - 3\eta\xi^2) + \\ +(\kappa_1 \cos Nt - \kappa_2 \sin Nt)(\xi^3 - 3\xi\eta^2)\right] + \varepsilon^3\left[H_3^{(2)}(\xi,\eta,t) + H_4^{(1)}(\xi,\eta,t)\right] + O(\varepsilon^4) \tag{2.3}$$

where κ_1 and κ_2 are constants. We shall assume that $\kappa_1^2 + \kappa_2^2 \neq 0$.

We now replace ξ and η by new canonical conjugate variables φ and r through the canonical transformation

$$\xi = \sqrt{2r}\sin[\varphi + (Nt - \varphi_0)/3], \quad \eta = \sqrt{2r}\cos[\varphi + (Nt - \varphi_0)/3] \\ \sin\varphi_0 = \kappa_1(\kappa_1^2+\kappa_2^2)^{-1/2}, \quad \cos\varphi_0 = \kappa_2(\kappa_1^2+\kappa_2^2)^{-1/2} \tag{2.4}$$

The new Hamiltonian is

$$F = \varepsilon^2(cr^2 + \kappa r^{3/2}\cos 3\varphi) + \varepsilon^3[F_3(\varphi, r, t) + F_4(\varphi, r, t)] + O(\varepsilon^4) \quad (2.5)$$

where $\kappa = 2[2(\kappa_1^2 + \kappa_2^2)]^{1/2}$; F_3 and F_4 are the functions $H_3^{(2)}$ and $H_4^{(1)}$ of (2.3) after the substitution (2.4); these functions are 6π-periodic in t.

For convenience, we shall use one more canonical transformation

$$r = \kappa^2 c^{-2}\rho, \quad \varphi = s(\theta - \pi/6) + \pi/6 \quad (s = \operatorname{sign} c) \quad (2.6)$$

and introduce a new independent variable $\tau = \varepsilon^2\kappa^2|c|^{-1}t$. The equations of motion in the new variables are

$$d\theta/d\tau = \partial\gamma/\partial\rho, \quad d\rho/d\tau = -\partial\gamma/\partial\theta \quad (2.7)$$

$$\gamma = \gamma_0(\theta, \rho) + \varepsilon\gamma_1(\theta, \rho, \tau) + O(\varepsilon^2) \quad (2.8)$$

$$\gamma_0 = \rho^2 + \rho^{3/2}\cos 3\theta, \quad \gamma_1 = c^3\kappa^{-4}(F_3 + F_4) \quad (2.9)$$

where F_3 and F_4 are the functions of (2.5) after the substitution (2.6). The function γ_1 may be written as

$$\gamma_1 = \rho^{3/2}\left(\alpha_1^{(3)}\cos\theta + \beta_1^{(3)}\sin\theta + \alpha_3^{(3)}\cos 3\theta + \beta_3^{(3)}\sin 3\theta\right) +$$

$$+\rho^2\left(\alpha_0^{(4)} + \alpha_2^{(4)}\cos 2\theta + \beta_2^{(4)}\sin 2\theta + \alpha_4^{(4)}\cos 4\theta + \beta_4^{(4)}\sin 4\theta\right)$$

$$\alpha_j^{(m)} = \sum_{k=-\infty}^{+\infty}{}' a_k^{(j,m)}\exp(\tfrac{3}{2}ik\lambda\tau), \quad \beta_j^{(m)} = \sum_{k=-\infty}^{+\infty}{}' b_k^{(j,m)}\exp(\tfrac{3}{2}ik\lambda\tau) \quad (2.10)$$

where $a_k^{(j,m)}$, $b_k^{(j,m)}$ are constant Fourier coefficients and

$$\lambda = (2/9)|c|\kappa^{-2}\varepsilon^{-2} \quad (2.11)$$

The prime on the summation symbol in (2.10) indicates that the summation is performed only for $k \neq 0$.

3. MOTION OF THE UNPERTURBED SYSTEM

Putting $\varepsilon = 0$ in (2.7), we obtain the equations of motion of the "unperturbed" system

$$d\theta/d\tau = 2\rho + \tfrac{3}{2}\rho^{1/2}\cos 3\theta, \quad d\rho/d\tau = 3\rho^{3/2}\sin 3\theta \quad (3.1)$$

which are satisfied by the Hamiltonian γ_0 of (2.9).

The unperturbed system has a first integral

$$\rho^2 + \rho^{3/2}\cos 3\theta = h \quad (h = \text{const}) \quad (3.2)$$

The phase portrait of system (3.1) has periodic $2\pi/3$ with respect to θ; it is shown in Fig. 3. In the x_1, x_2 plane, where x_1 and x_2 are defined by (1.3), the phase portrait is shown in Fig. 2. When $h < -27/256$, the motion is impossible. The value $h = -27/256$ corresponds to an equilibrium position $\rho_* = 9/16$: $\theta_* = (2l-1)\pi/3$ $(l = 1, 2, 3)$. If $-27/256 < h < 0$ (oscillatory domain),

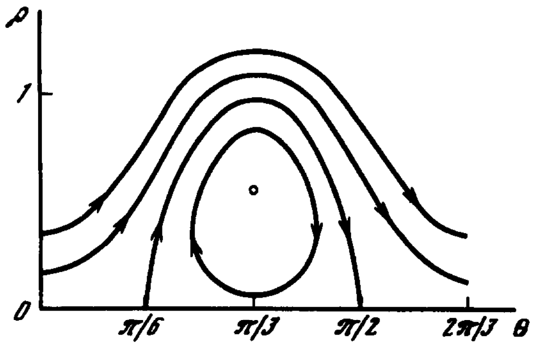

Fig. 3.

the variables ρ and θ vary periodically in the neighbourhood of their equilibrium values ρ_* and θ_*. When that happens $\rho_2 \leqslant \rho \leqslant \rho_1$ where ρ_1, $\rho_2 (0 < \rho_2 < 9/16 < \rho_1 < 1)$ are the real roots of the equation $\rho^2 - \rho^{3/2} = h$. In the oscillatory domain Eqs (3.1) are integrable in terms of elliptic functions. Introducing the notation

$$2\alpha = 1 - \rho_1 - \rho_2, \; f = -3\alpha(\rho_1 + \rho_2) - 2h, \; \beta^2 = f - (\alpha - \rho_1)(\alpha - \rho_2)$$
$$g = \beta^2(\rho_1 - \rho_2)^2, \; \mu = (f^2 + g)^{1/4}, \; 2k^2 = 1 - f/\mu^2 \tag{3.3}$$
$$\delta = \{[(\alpha - \rho_2)^2 + \beta^2]/[(\alpha - \rho_1)^2 + \beta^2]\}^{1/2}$$

and assuming that ρ has its minimum value ρ_2 when $\tau = 0$, we obtain

$$\rho(\tau) = \frac{\rho_1\delta + \rho_2 - (\rho_1\delta - \rho_2)\operatorname{cn}(3\mu\tau)}{1 + \delta + (1 - \delta)\operatorname{cn}(3\mu\tau)} \tag{3.4}$$

and $\theta(\tau)$ may be determined from (3.2). The symbol *cn* in (3.4) denotes the elliptic cosine function; the modulus k of the elliptic functions was defined in (3.3).

The oscillation frequency ω_1 is given by

$$\omega_1 = 3\pi\mu / (2K(k)) \tag{3.5}$$

where $K(k)$ is the complete elliptic integral of the first kind. Letting $h \to -27/256$, we obtain the frequency $\omega_1 = 9\sqrt{(3)}/8$ of small linear oscillations in the neighbourhood of the equilibrium point ρ_*, θ_*. As $h \to 0$ we have $\omega_1 \to 0$, whence

$$\omega_1 = -ah^{1/3} + O\left(|h|^{2/3}\right), \; a = 3^{3/4}\pi / \{2K[(\sqrt{6} - \sqrt{2})/4]\} = 3{,}94 \tag{3.6}$$

When $h > 0$ (the domain of rotations) $\rho(\tau)$ is given as before by formulae (3.3) and (3.4), except that now ρ_2 is a real root of the equation $\rho^2 + \rho^{3/2} = h$. The angle θ increases monotonically with τ and varies by 2π in "time" $6\pi/\omega_1$. The average frequency ω_2 of rotations is given by

$$\omega_2 = \pi\mu/(2K(k)) \tag{3.7}$$

As $h \to 0$ we have $\omega_2 \to 0$ and

$$\omega_2 = ah^{1/3}/3 + O(h^{2/3}) \tag{3.8}$$

where a is the constant of (3.6). If $h \ll 1$, we have $\omega_2 = 2h^{1/2} + O(h^{-1/2})$.

If $h = 0$, then either the system is in the equilibrium state $\rho_* = 0$, or its trajectories are doubly asymptotic to that point. The maximum value of ρ on these trajectories is unity. Suppose that ρ takes this value when $\tau = 0$. Then along the doubly asymptotic trajectories

$$\rho = 4/(4+9\tau^2), \quad \theta = [(2l-1)\pi + \mathrm{arctg}(3\tau/2)]/3 \quad (l = 1,2,3) \tag{3.9}$$

These trajectories are the separatrices of the domains of oscillatory and rotational motion in Figs 2 and 3. Henceforth we shall denote each separatrix by S_0.

In the domains of oscillations and rotations of the unperturbed problem with Hamiltonian γ_0 we can introduce action-angle variables I and w. The action $I(h)$ is defined in the oscillatory domain by the integral

$$I = (2\pi)^{-1} \oint \rho d\theta \tag{3.10}$$

where ρ is the function of θ and h is defined by (3.2), and the integral is evaluated along a closed phase curve of Fig. 3 surrounding the equilibrium position ρ_{*}, θ_{*}. The function inverse to (3.10) gives an expression for the unperturbed Hamiltonian in terms of the action variable: $\gamma_0 = h(I)$.

In the limiting case $h = -27/256$ (equilibrium) and $h = 0$ (separatrix) I equals 0 and 1/12, respectively. Figure 4 shows a graph of the function $h(I)$; graphs of $\omega_1 = dh/dI$ and $d\omega_1/dI$ plotted against h are also shown.

Near the separatrix we have the following expansions

$$h(I) = -2^{1/2} a^{3/2} (1-12I)^{3/2}/108 + O((1-12I)^2)$$
$$d\omega_1/dI = -2^{1/2} a^{3/2} (1-12I)^{-1/2} + O(1) = a^2 h^{-1/3}/3 + O(1)$$

In the domain of rotations the action I is defined by the integral

$$I = (2\pi)^{-1} \int_0^{2\pi} \rho d\theta$$

where $\rho = \rho(\theta, h)$ is found from (3.2). As $h \to 0$ we have $I \to 1/4$; if $h \ll 1$, then $I = h^{1/2} + O(h^{-1/2})$. Graphs of the functions $h(I)$, $\omega_2 = dh/dI$ and $d\omega_2/dI$ are shown in Fig. 5. Near the separatrices

$$h(I) = 2^{1/2} a^{3/2} (4I-1)^{3/2}/108 + O((4I-1)^2)$$
$$d\omega_2/dI = 2^{1/2} a^{3/2} (4I-1)^{-1/2}/9 + O(1) = a^2 h^{-1/3}/27 + O(1)$$

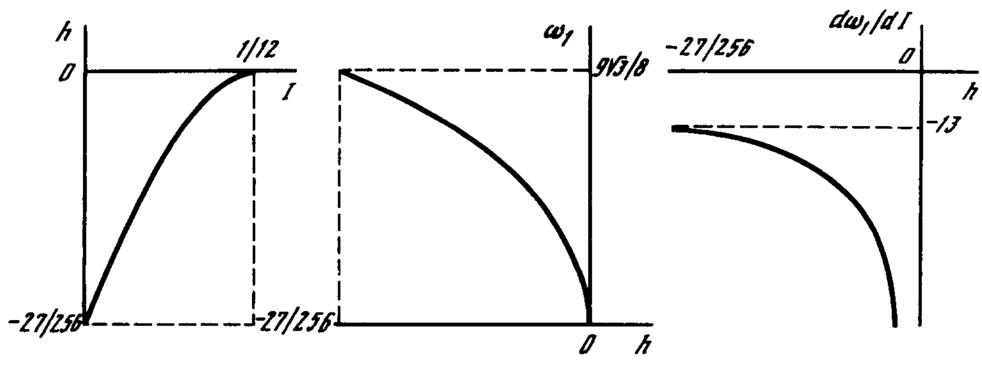

Fig. 4.

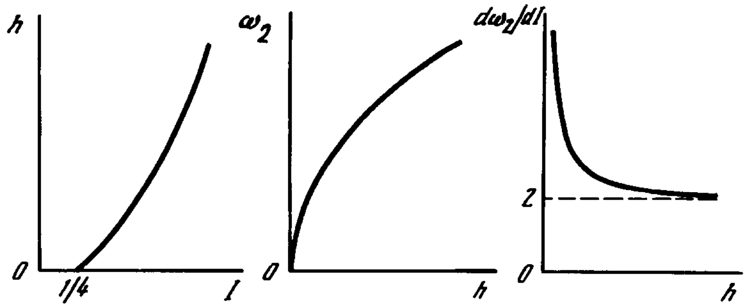

Fig. 5.

The Hamiltonian γ_0 of the unperturbed motion is non-degenerate in the domains of both oscillatory and rotational motion: the derivative d^2h/dI^2 does not vanish—it is negative in the oscillatory domain and positive in the rotational domain.

4. PERTURBED MOTION IN THE DOMAINS OF OSCILLATION AND ROTATION

Let us assume now that ε is small but non-zero. When $\varepsilon \neq 0$ the equilibrium positions $\rho_* = 9/16$, $\theta_* = (2l-1)\pi/3$ $(l = 1, 2, 3)$ are replaced by 6π-periodic motion (with respect to the original independent variable t) which depends analytically on ε. This follows from Poincaré's theory of periodic motion in quasi-linear systems [7], since the frequency of small linear oscillations of the unperturbed system (3.1) in the neighbourhood of the equilibrium ρ_*, θ_* is $\varepsilon^2 9\sqrt{(3)}\kappa^2/(8|c|)$ and if ε is small enough this cannot be a multiple of the frequency of periodic perturbation, which is 1/3.

Computations show that the Hamiltonian (2.8), normalized in the neighbourhood of the 6π-periodic motion, may be expanded up to second-order terms inclusive as follows:

$$\gamma = \Omega_1 I + \Omega_2 I^2$$

where $\Omega_1 = 9\sqrt{(3)}/8 + f_1(\varepsilon)$, $\Omega_2 = -13/2 + f_2(\varepsilon)$, and $f_i(\varepsilon)$ tends to zero as $\varepsilon \to 0$. Since Ω_2 does not vanish for $0 < \varepsilon \ll 1$, it follows that the periodic motion in question is stable in Lyapunov's sense for sufficiently small ε [8, 9].

The perturbed motion in the neighbourhood of the separatrices (3.9) will be considered later. But if we exclude a small neighbourhood of the separatrices, the Hamiltonian (2.8) will be analytic as a function of the action-angle variables, and moreover, as already remarked, its unperturbed part will satisfy non-degeneracy conditions in both the oscillatory and rotational domains. Therefore [10], most of the phase trajectories lying in these domains give rise to quasi-periodic motions when ε is small.

In the space spanned by θ, ρ, t (or x_1, x_2, t), planes $t = \text{const}$ separated by distances that are multiples of 6π may be identified. On the surface of the "section" $t = 0$ the above-mentioned conditionally periodic trajectories coincide with the closed phase curves of Fig. 3 (or Fig. 2) when $\varepsilon = 0$. For small but non-zero ε, most of these curves remain closed, experiencing only slight deformations (together with ε). By [11], the Lebesgue measure of the phase curves that "rupture" when $\varepsilon \neq 0$ is exponentially small; in the problem being considered here it is of the same order of magnitude as $\exp(-c_1\varepsilon^{-2})$ $(c_1 > 0 = \text{const})$.

5. PERTURBATION OF DOUBLY ASYMPTOTIC TRAJECTORIES

The homoclinic doubly asymptotic trajectory-separatrices (3.9) may be viewed as curves formed by two asymptotic trajectories S_0^+ and S_0^-: for S_0^+ the trajectory approaches the origin asymptotically as $t\to+\infty$, and for S_0^- it does so as $t\to-\infty$. The trajectories S_0^+ and S_0^- coincide and form the separatrix S_0 in the phase plane of the unperturbed system (3.1).

When $\varepsilon\neq 0$ the role of such a separating curve is played by a certain curve $S\varepsilon$. When $\varepsilon=0$ it coincides with S_0, but for $0<\varepsilon\ll 1$ it is generally the union of two branches S_ε^+ and S_ε^-. A point of the phase plane belongs to $S_\varepsilon^+(S_\varepsilon^-)$ if the trajectory of the perturbed system starting at that point when $t=0$ approaches the origin as $t\to+\infty$ $(t\to-\infty)$. When S_ε^+ and S_ε^- do not coincide, one says that the separatrix has split. The phenomenon was already observed by Poincaré [12].

Consider the following 2π-periodic function of α

$$J(\alpha)=\int_{-\infty}^{+\infty}(\gamma_0,\gamma_1)d\tau \tag{5.1}$$

where $(\gamma_0,\ \gamma_1)$ denotes the Poisson bracket, calculated along the double asymptotic trajectory (3.9), with the "time" τ, which occurs explicitly in γ_1, replaced by $\tau+2\alpha/(3\lambda)$. The function (5.1) depends on a parameter λ and on a number l ($l=1$, 2, 3) indicating which of the trajectories (3.9) is being considered. Functions of this type are used in Mel'nikov's method [13], which gives an estimate in powers of ε for the magnitude of the splitting of separatrices. In the simplest case, when the integral (5.1) is independent of ε, if $J(\alpha)\not\equiv 0$ splitting occurs, and if the equation $J(\alpha)=0$ has a simple root, then S_ε^+ and S_ε^- intersect an infinite number of times. This behaviour of the doubly asymptotic trajectories implies that the motion in the perturbed problem becomes chaotic. When the integral (5.1) depends on ε the method of [13] may be difficult to apply.

The evaluation and analysis of the integral (5.1) present a rather complex problem. It is somewhat simplified by the fact that one is investigating the behaviour of the perturbed system for small ε, so that (see (2.11)) for our present purposes it will suffice to know the behaviour of (5.1) as $\lambda\to+\infty$.

Taking into consideration that $\rho^{3/2}=-\cos 3\theta$ on the trajectories (3.9), we calculate the Poisson bracket $(\gamma_0,\ \gamma_1)$ and then replace the variable of integration τ in (5.1) by the variable θ defined in the second equality of (3.9). We obtain

$$J(\alpha)=\int_{\theta_1}^{\theta_2}\cos^2 3\theta f(\theta)d\theta$$

$$\theta_1=(4l-3)\pi/6,\ \theta_2=(4l-1)\pi/6$$

$$f(\theta)=-3\beta_3^{(3)}+(\alpha_2^{(4)}-4\alpha_1^{(3)}+2\alpha_4^{(4)})\sin 2\theta-(\beta_2^{(4)}+4\beta_1^{(3)}-2\beta_4^{(4)})\cos 2\theta+$$
$$+(2\alpha_4^{(4)}-5\alpha_1^{(3)}+5/2\,\alpha_2^{(4)})\sin 4\theta-(2\beta_4^{(4)}-5\beta_1^{(3)}-5/2\,\beta_2^{(4)})\cos 4\theta+$$
$$+6(\alpha_0^{(4)}-\alpha_3^{(3)})\sin 6\theta+6\beta_3^{(3)}\cos 6\theta+7/2\,\alpha_2^{(4)}\sin 8\theta-$$
$$-7/2\,\beta_2^{(4)}\cos 8\theta+4\alpha_4^{(4)}\sin 10\theta-4\beta_4^{(4)}\cos 10\theta$$

$$\alpha_j^{(m)}=\sum_{k=-\infty}^{+\infty}{}' a_k^{(j,m)}\exp(ik\alpha)\exp(ik\lambda\,\mathrm{tg}\,3\theta),\ \beta_j^{(m)}=\sum_{k=-\infty}^{+\infty}{}' b_k^{(j,m)}\exp(ik\alpha)\exp(ik\lambda\,\mathrm{tg}\,3\theta)$$

Consequently, evaluation of (5.1) reduces to the evaluation of integrals of the form

$$J_1^{(n,k)}=\int_{\theta_1}^{\theta_2}\cos^2 3\theta\sin 2n\theta\exp(ik\lambda\,\mathrm{tg}\,3\theta)d\theta,\ J_2^{(n,k)}=\int_{\theta_1}^{\theta_2}\cos^2 3\theta\cos 2n\theta\exp(ik\lambda\,\mathrm{tg}\,3\theta)d\theta \tag{5.2}$$

for $n=0, 1, \ldots, 5$ and integer k other than zero. After changing variables $z=3\theta-(2l-1)\pi$ we obtain the following expressions for the integrals (5.2)

$$J_1^{(n,k)}=i(uI^{(n,k)}-u^{-1}I^{(-n,k)})/3,\ J_2^{(n,k)}=(uI^{(n,k)}+u^{-1}I^{(-n,k)})/3 \tag{5.3}$$
$$u=\exp[i2n(l-2)\pi/3]$$
$$I^{(n,k)}=\int_0^{\pi/2}\cos^2 z\cos(k\lambda\,\mathrm{tg}\,z-\frac{2}{3}nz)dz$$

The last integral is expressed in terms of the Whittaker function [14]. When $\lambda \to +\infty$ it can be represented in the form

$$I^{(n,k)} = \frac{\pi(2|k|\lambda)^r}{8\Gamma(1+r)}\exp(-|k|\lambda)(1+O(\lambda^{-1})) \tag{5.4}$$

where $r = sn/3+1$, $S = \operatorname{sign}k$ and Γ is the gamma function.

Let q $(q>0)$ be the order of the lowest harmonic in the Fourier series (2.10). Then, using (5.2)–(5.4), we obtain that as $\lambda \to +\infty$

$$J(\alpha) = \chi\sin(q\alpha+\delta_1)\lambda^p\exp(-q\lambda)(1+O(\lambda^{-\delta_2})) \tag{5.5}$$

where χ, δ_1, δ_2 are constants, χ being independent of l, the number p is one of $5/3$, $6/3$, $7/3$, $8/3$, and $\delta_2 \geqslant 1/3$.

The quantity $J(\alpha)$ is exponentially small as $\varepsilon \to 0$. Direct application of the method of [13] would therefore be illegitimate in this case, as it has not been shown to be applicable for problems with exponentially small splitting of separatrices. However, it is obviously correct to state that splitting of separatrices occurs in the case of the general position, and one obtains stochastic motion in the neighbourhood of the separatrices. We shall now present a non-rigorous discussion of some questions concerning perturbed motion near the separatrices.

6. MOTION IN THE NEIGHBOURHOOD OF THE SEPARATRICES

We will first consider a useful transformation in an arbitrary nearly integrable Hamiltonian system with one degree of freedom. Write the Hamiltonian in the form (2.8). When $\varepsilon = 0$ the general solution of the differential equations (2.7) may be written in the form

$$\rho = \rho(\tau+\sigma, h), \quad \theta = \theta(\tau+\sigma, h) \tag{6.1}$$

where σ and h are arbitrary constants: σ is the value of the "phase" $\tau+\sigma$ at $\tau = 0$ and h the energy integral constant $\gamma_0(\theta, \rho) = h$.

It can be shown that if $\varepsilon \neq 0$ and the equalities (6.1) are treated as a change of variables θ, $\rho \to \sigma$, h, then this change of variables is a canonical univalent transformation. In the transformed system, σ and h play the role of coordinate and momentum, respectively, and the new Hamiltonian will be the function $\gamma - \gamma_0 = \varepsilon\gamma_1 + \ldots$, with ρ and θ replaced by the right-hand sides of (6.1)

$$dh/d\tau = -\varepsilon\partial\gamma_1/\partial\sigma - \ldots, \quad d\sigma/d\tau = \varepsilon\partial\gamma_1/\partial h + \ldots \tag{6.2}$$

It can further be shown that the first-order terms in ε in the first of equalities (6.2) may be written as $\varepsilon(\gamma_0, \gamma_1)$, where the Poisson brackets are evaluated for the values of ρ and θ in (6.1).

To investigate the perturbed motion in the neighbourhood of the separatrix, we construct a separatrix mapping for the system with Hamiltonian (2.8). Separatrix mappings are effective tools for establishing the conditions for the onset of stochastic motion and for analysing its properties. They have been widely used in many problems of mechanics and physics [15–19].

To obtain a separatrix mapping we use Eqs (6.2). Let h_0 and σ_0 be the values of h and σ at $\tau = 0$, and let $|h_0| \ll 1$. Let us determine the increment to h and σ over one cycle of the motion; the duration of the cycle equals the period of the oscillations or one third of the "period" of rotations in a small neighbourhood of the separatrix.

Sufficiently close to the separatrix, one can put $h = 0$ on the right-hand side of the first of Eqs (6.2) and approximate the increment $h_1 - h_0$ of h by integrating between infinite limits. We obtain

$$h_1 = h_0 + \varepsilon \int_{-\infty}^{+\infty} (\gamma_0, \gamma_1) d\tau \tag{6.3}$$

where the Poisson bracket is evaluated on the separatrix $\rho = \rho(\tau + \sigma_0, 0)$, $\theta = \theta(\tau + \sigma_0, 0)$. Applying the translation $\tau \to \tau - \sigma_0$ in (6.3), we obtain

$$h_1 = h_0 + \varepsilon G(\sigma_0)$$

$$G(\sigma_0) = \int_{-\infty}^{+\infty} (\gamma_0(\theta, \rho), \gamma_1(\theta, \rho, \tau - \sigma_0)) d\tau \tag{6.4}$$

where θ and ρ are the functions given in (3.9).

It follows from (3.6) and (3.8) that the duration of one cycle of motion near the separatrix is $2\pi a^{-1} |h|^{-1/3}$. Therefore

$$\sigma_1 = \sigma_0 + 2\pi a^{-1} |h|^{-1/3} + \varepsilon g(\sigma_0, h_0) \tag{6.5}$$

The last term in (6.5) may be obtained by integrating the right-hand side of the second equation of (6.2) between infinite limits, as was done to evaluate the function $G(\sigma_0)$ of (6.4). But it is easier to derive it from the condition that the mapping σ_0, $h_0 \to \sigma_1$, h_1 defined by (6.4) and (6.5) should be canonical. Indeed, equating the first-degree terms in ε on the right of the canonicity condition

$$\frac{\partial \sigma_1}{\partial \sigma_0} \frac{\partial h_1}{\partial h_0} - \frac{\partial \sigma_1}{\partial h_0} \frac{\partial h_1}{\partial \sigma_0} = 1$$

to zero, we at once obtain

$$g = 2\pi a^{-1} G(\sigma_0) d|h_0|^{-1/3} / dh_0$$

Ignoring terms of order ε^2, we can now write (6.5) as

$$\sigma_1 = \sigma_0 + 2\pi a^{-1} |h_1|^{-1/3} \tag{6.6}$$

If we also apply the change of variables $\sigma = -2\alpha/(3\lambda)$, we can write the separatrix mapping as

$$h_1 = h_0 + \varepsilon J(\alpha_0), \quad \alpha_1 = \alpha_0 - 3\pi a^{-1} |h_1|^{-1/3} \tag{6.7}$$

where J is the integral (5.1), and the values of λ and α are given by (2.11) and (3.6).

Noting that when $\lambda \gg 1$ the integral $J(\alpha)$ admits of the representation (5.5) and applying the change of variables $\beta = q\alpha + \delta_1$, we finally obtain the following approximate representation of the separatrix mapping for $0 < \varepsilon \ll 1$

$$h_1 = h_0 + \varepsilon \chi \lambda^p e^{-q\lambda} \sin \beta_0, \beta_1 = \beta_0 - 3\pi q a^{-1} \lambda |h_1|^{-1/3} \tag{6.8}$$

Values of β differing from one another by a multiple of 2π will be identified.

Let us consider the fixed points of the mapping (6.8). At fixed points, h takes the values

$$h = h_* = \pm(3q\lambda / (2an))^3 \quad (n = 1, 2, 3 \ldots) \tag{6.9}$$

and β is either $\beta^{(1)} = 0$ or $\beta^{(2)} = \pi$. The upper sign in (6.9) corresponds to rotations and the lower sign to oscillations.

Analysis shows that the fixed points h_*, $\beta^{(i)}$ are stable in the linear approximation if it is true that

$$\left|1-\varepsilon(-1)^i\frac{\pi\chi q}{2ah_*|h_*|^{1/3}}\lambda^{p+1}e^{-q\lambda}\right|<1\ (i=1,2)$$

If the sign of this inequality is reversed, we get instability. In particular, if $|h_*|$ is small enough

$$|h_*|<\left(\frac{\pi|\chi|q}{4a}\right)^{3/4}\varepsilon^{3/4}\lambda^{3(p+1)/4}e^{-3q\lambda/4} \tag{6.10}$$

then all the fixed points of the separatrix mapping are unstable.

In the perturbed problem, stochastic motions arise for arbitrarily small (but non-zero) values of ε. A stochastic layer forms fairly close to the separatrix. We will estimate its width using Chirikov's method [15–17]. To that end, consider the linearized separatrix mapping (6.8) in the neighbourhood of the values of h_* corresponding to its fixed points. Putting

$$h=h_*+\frac{ah_*|h_*|^{1/3}}{\pi q\lambda}P$$

and linearizing (6.8) with respect to P, we obtain the mapping

$$P_1=P_0+K\sin\beta_0,\quad \beta_1=\beta_0+P_1 \tag{6.11}$$

where K is the stochasticity parameter, defined by the formula

$$K=\varepsilon\frac{\pi\chi q}{ah_*|h_*|^{1/3}}\lambda^{p+1}e^{-q\lambda} \tag{6.12}$$

In the theory of stochastic motion, the mapping (6.11) is called a standard mapping [16].

An estimate for the width of the stochastic layer may be obtained [15–17] from the inequality $|K|>1$. Using (6.12), we can write this inequality in the form (6.10) by formally replacing $4a$ by a.

This approximate estimate for the half-width of the stochastic layer may be improved [16] by a careful study of the properties of the standard mapping (6.11), but for small ε the order of magnitude of the half-width remains equal to $\varepsilon^{-b}\exp(-c_2\varepsilon^{-2})$ $(b=3(2p+1)/4,\ c_2=|c|q\kappa^{-2}/6)$ and is the same as that of the expression on the right of the inequality.

In conclusion, we note that, regardless of the instability of the equilibrium position $x=y=0$ of the initial system with Hamiltonian (1.1), trajectories beginning sufficiently close to the origin will remain throughout the motion in a bounded neighbourhood of the point $x=y=0$. The above investigation has shown that for such trajectories ρ will never exceed a value close to unity. In the x, y plane, therefore, a trajectory $x(t), y(t)$ will always remain inside a circle of radius $2^{1/2}\kappa|c|^{-1}\varepsilon(1+\psi_1(x(0),\ y(0))+\psi_2(\varepsilon))$, where $\psi_1\to 0$ as $x^2(0)+y^2(0)\to 0$ and $\psi_2\to 0$ as $\varepsilon\to 0$.

I wish to thank A. I. Neishtadt for useful discussions.

The research reported here was supported financially by the Russian Fund for Fundamental Research (93-013-16257) and the International Scientific Fund.

REFERENCES

1. MARKEYEV A. P., *Libration Points in Celestial Mechanics and Space Dynamics*. Nauka, Moscow, 1978.
2. MARKEYEV A. P., and SHCHERBINA G. A., The motions of a satellite asymptotic to its eccentric oscillations. *Izv. Akad. Nauk SSSR. MTT* 4, 3–10, 1987.

3. MARKEYEV A. P., Resonances and asymptotic trajectories in Hamiltonian systems. *Prikl. Mat. Mekh.* **54**, 2, 207–212, 1990.
4. POINCARÉ H., *New Methods of Celestial Mechanics. Collected Papers*, Vol. 2. Nauka, Moscow, 1972.
5. GIACAGLIA G. E. O., *Perturbation Methods in Non-linear Systems.* Springer, New York, 1972.
6. BIRKHOFF G. D., *Dynamical Systems.* American Mathematical Society, New York, 1927.
7. MALKIN I. G., *Some Problems of the Theory of Non-linear Oscillations.* Moscow, Gostekhizdat, 1956.
8. ARNOL'D V. I., Small denominators and problems of the stability of motion in classical and celestial mechanics. *Uspekhi Mat. Nauk* **18**, 6, 91–192, 1963.
9. MOSER E., *Lectures on Hamiltonian Systems*. American Mathematical Society, Providence, RI, 1968.
10. ARNOL'D V. I., KOZLOV V. V. and NEISHTADT A. I., *Mathematical Aspects of Classical and Celestial Mechanics. Itogi Nauki i Tekhniki. Ser. Sovremennye Problemy Matematiki, Fundamental'nye Napravleniya*, Vol. 3. VINITI, Moscow, 1985.
11. NEISHTADT A. I., Estimates in Kolmogorov's theorem on the conservation of conditionally periodic motions. *Prikl. Mat. Mekh.* **45**, 6, 1016–1025, 1981.
12. POINCARÉ H., On the three-body problem and the equations of dynamics. In *Collected Papers*, Vol. 2, pp. 357–444. Nauka, Moscow, 1972.
13. MEL'NIKOV V. K., On the stability of the centre in time-periodic perturbations. *Trudy Moskov. Mat. Obschch.* **12**, 3–52, 1963.
14. GRADSHTEIN I. S. and RYZHIK I. M., *Tables of Integrals, Sums, Series and Products.* Fizmatgiz, Moscow, 1962.
15. CHIRIKOV B. V., *Non-linear Resonance.* Novosibirsk. Gos. Univ., Novosibirsk, 1977.
16. CHIRIKOV B. V., *Interaction of Non-linear Resonances.* Novosibirsk. Gos. Univ., Novosibirsk, 1978.
17. CHIRIKOV B. V., A universal instability of many-dimensional oscillator systems. *Phys. Reports* **52**, 5, 265–379, 1979.
18. LICHTENBERG A. J. and LIBERMAN M. A., *Regular and Stochastic Motion.* Springer, New York, 1983.
19. ZASLAVSKII G. M. and SAGDEYEV R. Z., *Introduction to Non-linear Physics.* Nauka, Moscow, 1988.
20. ZASLAVSKII G. M. and CHIRIKOV B. V., Stochastic instability of non-linear oscillations. *Uspekhi Mat. Nauk* **105**, 1, 3–39, 1971.

Translated by D.L.